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ON A NEW CLASS OF MOTIONS OF A SYSTEM OF HEAVY HINGED RIGID BODIES*

G.V. GORR and V.N. RUBANOVSKII

A chain of heavy rigid bodies with one or both ends fixed is considered. If the arrangement is such that each hinge is at the point of intersection of the perpendiculars to the planes of circular section of the central gyratory ellipsoids of adjacent bodies, the equations of motion of each body can be shown to admit of a Hess invariant relation /1/. A geometrical interpretation of the motion is given when such invariant relations exist. The simplest classes of motion are considered, and the conditions for semiregular precessions to exist are indicated.

1. We consider the motion of a system of heavy hinged gyrostats $S_1, S_2, \ldots, S_n$ $(n \geqslant 1)$ in a uniform gravitational field, in which the first gyrostat S_1 has a fixed point O_1, while the rest are successively interconnected as a chain by means of ideal spherical hinges $O_2, \ldots, O_n$ in such a way that the centre of mass of each body and the hinges connecting it to the adjacent bodies (or to the fixed point) are collinear. The system is then a chain with one fixed end. We can also consider a similar chain with two fixed ends, in which case we assume that the last body S_n has a fixed point O_{n+1}, and that its centre of mass and the points O_n and O_{n+1} are collinear.

We introduce the fixed system of coordinates O_1xyz, with z axis having the unit vector $\mathbf{v}$ directed vertically upwards, and the moving systems of coordinates $C_ix_1^ix_2^ix_3^i$ with origins at the centres of mass C_i of the bodies and axes directed along their principal central axes of inertia. Here and throughout, $i = 1, \ldots, n$.

Let Θ^i be the central tensor of inertia of the i-th body with diagonal elements $J_1^i < J_2^i < J_3^i$ (in the $C_ix_1^ix_2^ix_3^i$ system of coordinates), while $\boldsymbol{\omega}$ and $\mathbf{k}^i$ are the vectors of the absolute instantaneous angular velocity of the i-th body and its gyrostatic momentum; $\mathbf{r}^i$ $\mathbf{r}^{i+1}$ $(\mathbf{r}^{i+1} = \lambda^i\mathbf{r}^i$, λ^i is a constant) are the radius vectors of the points O_i and O_{i+1} with respect to the point C_i; ω_j^i, k_j^i, e_j^i $(j = 1, 2, 3)$ are the projections of the vectors $\boldsymbol{\omega}^i$, $\mathbf{k}^i$, $\mathbf{r}^i$ onto the x_j^i axes. If the last body S_n has no fixed point, we shall assume that $\lambda^n = 0$.

Let the action of the body S_{i-1} on S_i be characterized by the force $\mathbf{R}^i$; then the action of S_{i+1} on S_i is characterized by the force $\mathbf{R}^{i+1}$.

Using the theorem on the variation of the angular momentum in relative motion about the centre of mass, we obtain for the i-th body the equation

$$\Theta^i \cdot \boldsymbol{\omega}^{\cdot i} + \boldsymbol{\omega}^i \times (\Theta^i \cdot \boldsymbol{\omega}^i + \mathbf{k}^i) = \mathbf{r}^i \times (\mathbf{R}^i - \lambda^i \mathbf{R}^{i+1})$$

Projecting this equation onto the x_1^i, x_2^i, x_3^i axes, we obtain the equations

$$J_1^i\omega^{\cdot i} + (J_3^i - J_2^i)\,\omega_2^i\omega_3^i + k_3^i\omega_2^i - k_2^i\omega_3^i = e_2^i(R_3^i - \lambda^i R_3^{i+1}) - e_3^i(R_2^i - \lambda^i R_2^{i+1}) \quad (123) \tag{1.1}$$

Assume that we have the conditions

$$e_2^i = 0, \quad k_2^i = 0 \tag{1.2}$$

On multiplying the first and third equations of (1.1) by e_1^i and e_3^i respectively, and adding term by term, we obtain the relations

$$dV^i/dt + \omega_2^i[(J_2^i - J_1^i)\,\omega_1^ie_3^i + (J_3^i - J_2^i)\,\omega_3^ie_1^i + k_3^ie_1^i - k_1^ie_3^i] = 0$$
$$V^i = J_1^i\omega_1^ie_1^i + J_3^i\omega_3^ie_3^i + k_3^ie_1^i - k_1^ie_3^i$$

from which, under the conditions

$$\frac{(J_2^i - J_1^i)\,e_3^i}{J_1^ie_1^i} = \frac{(J_3^i - J_2^i)\,e_1^i}{J_3^ie_3^i} \tag{1.3}$$

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we obtain the invariant relations

$$V^i = 0 \tag{1.4}$$

In short, under conditions (1.2), (1.3), our system of bodies admits of the system of invariant relations (1.4). Another statement of this result is possible. The chain of bodies admits of the system of invariant relations (1.4) if the centre of mass C_i of each body S_i and the hinges O_i, O_{i+1} connecting it with the adjacent bodies (or with the fixed point) lie on the perpendicular to the circular section of its central gyrational ellipsoid, and the gyrostatic moment $\mathbf{k}^i$ lies in the plane perpendicular to this circular section.

Hence, for $n = 1$, we obtain Sretenskii's invariant relation /2/ for the integrability of the equations of motion of a heavy gyrostat, and from this, under the auxiliary conditions $k_1^1 = k_3^1 = 0$, Hess's invariant relation /1/. For two bodies, one of which is weightless, a relation of type (1.4) is given in /3/.

2. Let us analyse the motion of our chain when relations (1.4) hold. We shall assume for simplicity that all the $\mathbf{k}^i = 0$.

For each S_i we introduce the auxiliary orthogonal system of coordinates $C_iy_1^iy_2^iy_3^i$, in which the y_2^i axis coincides with the x_2^i axis, and the y_1^i axis passes through the point $O_iC_iO_{i+1}$.

We denote the unit vectors of this system by $\varepsilon_1^i, \varepsilon_2^i, \varepsilon_3^i$, putting

$$\varepsilon_1^i = C_iO_{i+1}/|C_iO_{i+1}|, \quad r^i = |O_iC_i|, \quad \rho^i = |C_iO_{i+1}|$$

Further, let Ω_j^i be the projections of the angular velocity vector $\boldsymbol{\omega}^i$ onto the y_j^i axes, and A_{11}^i, $A_{22}^i = J_2^i$, A_{33}^i, A_{13}^i be the non-zero components of the inertia tensor Θ^i in the $C_iy_1^iy_2^iy_3^i$ system of coordinates. The equations of motion are then

$$\Theta^i\cdot\boldsymbol{\omega}^{\cdot i} + \boldsymbol{\omega}^i \times \Theta^i\cdot\boldsymbol{\omega}^i = -\varepsilon_1 \times (r^i\mathbf{R}^i + \rho^i\mathbf{R}^{i+1}) \tag{2.1}$$

$$\boldsymbol{\omega}^i = \Omega_1^i\varepsilon_1^i + \Omega_2^i\varepsilon_2^i + \Omega_3^i\varepsilon_3^i, \quad \Theta^i = \begin{Vmatrix} A_{11}^i & 0 & -A_{13}^i \\ 0 & A_{22}^i & 0 \\ -A_{13}^i & 0 & A_{33}^i \end{Vmatrix} \tag{2.2}$$

In the $C_iy_1^iy_2^iy_3^i$ system, relations (1.4) can be written as

$$A_{11}^i\Omega_1^i - A_{13}^i\Omega_3 = 0 \tag{2.3}$$

Let φ_i be the angles of rotation of the bodies about the y_1^i axes, and let ψ_i, ϑ_i be the Euler angles defining the position of the step-line $O_1O_2\ldots O_{n+1}$ relative to the O_1xyz coordinate system. We then have for Ω_j^i the expressions

$$\Omega_1^i = \psi_i^{\cdot}\cos\vartheta_i + \varphi_i^{\cdot}, \quad \Omega_2^i = \psi_i^{\cdot}\sin\vartheta_i\sin\varphi_i + \vartheta_i^{\cdot}\cos\varphi_i \tag{2.4}$$
$$\Omega_3^i = \psi_i^{\cdot}\sin\vartheta_i\cos\varphi_i - \vartheta_i^{\cdot}\sin\varphi_i$$

The conditions that the straight lines through the points O_i, C_i, O_{i+1} be perpendicular to the planes of the circular sections of the central gyrational ellipsoids, lead to the relations

$$A_{11}^iA_{22}^i = A_{11}^iA_{33}^i - A_{13}^iA_{13}^i, \quad A_{22}^i = J_2^i \tag{2.5}$$

The expression for the kinetic energy of the system

$$T = \frac{1}{2}\sum_{i=1}^n (m_iv_i^2 + A_{11}^i\Omega_1^{i2} + A_{22}^i\Omega_2^{i2} + A_{33}^i\Omega_3^{i2} - 2A_{13}^i\Omega_1^i\Omega_3^i)$$

can be written, using (2.3) and (2.5), as

$$T = \frac{1}{2}\sum_{i=1}^n [m_iv^2_i + J_2^i(\Omega_2^{i2} + \Omega_3^{i2})] \tag{2.6}$$

(m_i and v_i are the mass and velocity of the centre of mass of the i-th body).

We conclude from this that the kinetic energy of the chain of rigid bodies is the same, in the light of relations (2.3), as the kinetic energy of a chain of rods with masses equal to the masses of the respective rigid bodies, and the central moments of inertia J_2^i.

From (2.6), using (1.4), we obtain the relation

$$T = T(\psi_1, \vartheta_1, \ldots, \psi_n, \vartheta_n, \psi_1^{\cdot}, \vartheta_1^{\cdot}, \ldots, \psi_n^{\cdot}, \vartheta_n^{\cdot}) \tag{2.7}$$

Now consider the potential energy Π of the chain of rigid bodies. It is the same as the potential energy of our chain of rods and is given by

$$\Pi = \Pi(\vartheta_1, \ldots, \vartheta_n) \tag{2.8}$$

For our chain of bodies the Lagrange function $L = T - \Pi$ is given, using (2.7) and (2.8) by

$$L = L(\psi_1, \vartheta_1, \ldots, \psi_n, \vartheta_n, \psi_1^{\cdot}, \vartheta_1^{\cdot}, \ldots, \psi_n^{\cdot}, \vartheta_n^{\cdot}) \tag{2.9}$$

In view of (2.4), we can write relations (2.3) as

$$A_{11}^{i}(\varphi_i^{\cdot} + \psi_i^{\cdot}\cos\vartheta_i) + A_{13}^{i}(\vartheta_i^{\cdot}\sin\varphi_i - \psi_i^{\cdot}\sin\vartheta_i\cos\varphi_i) = 0 \tag{2.10}$$

We conclude as a result that, when relations (2.3) hold, the motion of the chain of heavy rigid bodies is made up of two motions: 1) the motion of the step-line $O_1O_2 \ldots O_{n+1}$ as a chain of heavy rods with masses equal to the masses of the respective bodies, and central moments of inertia J_2^{i}, and 2) rotations of a body about the links of the step-line. The first motion is given by the Lagrange equations with Lagrange function (2.9), and the second by Eqs. (2.10).

3. We will indicate the simplest classes of motion of the chain of heavy bodies with one fixed end. Assume initially that

$$\vartheta_i = \vartheta_{i0}, \quad \vartheta_i^{\cdot} = 0, \quad \psi_i = \omega t, \quad \psi_i^{\cdot} = \omega = \text{const} \tag{3.1}$$

Here, ω is arbitrary, while the ϑ_{i0} are given by the equations which are obtained from the Lagrange equations with Lagrange function (2.9) after substituting in them the values (3.1).

Relations (3.1) describe uniform rotations about the vertical with angular velocity ω of the step-line $O_1O_2 \ldots O_n$ as a rigid body, all the links of which lie in the same vertical plane and make angles $\vartheta_i = \vartheta_{i0}$ with the vertical.

When (3.1) are satisfied, Eqs. (2.10) take the form

$$\varphi^{\cdot} = \omega(a_i + b_i\cos\varphi_i) \tag{3.2}$$
$$a_i = -\cos\vartheta_{i0}, \quad b_i = (A_{13}^{i}/A_{11}^{i})\sin\vartheta_{i0}$$

Hence we obtain the equations for $\varphi_i = \varphi_i(t)$:

$$\omega t + D_i = \begin{cases} \dfrac{2}{\sqrt{a_i^2 - b_i^2}} \operatorname{arctg} \dfrac{\Phi_i}{\sqrt{a_i^2 - b_i^2}}, & a_i^2 > b_i^2 \\[2ex] \dfrac{1}{\sqrt{b_i^2 - a_i^2}} \ln\left|\dfrac{\Phi_i - \sqrt{b_i^2 - a_i^2}}{\Phi_i + \sqrt{b_i^2 - a_i^2}}\right|, & a_i^2 < b_i^2 \end{cases} \tag{3.3}$$
$$\Phi_i = (a_i - b_i)\operatorname{tg} {}^1/_2\, \varphi_i$$

where D_i are constants of integration.

We see from (3.3) that the equation $\varphi_i = \varphi_i(t)$ gives a zig-zag line /4/ if $a_i^2 > b_i^2$, and a loxodrome /4/ if $a_i^2 < b_i^2$.

A second class of possible motions of the chain of bodies with one fixed end is as follows.

The Lagrange equations giving the motions of the chain have the solutions

$$\psi_i = 0, \ \psi_i^{\cdot} = 0, \ \vartheta_i = \vartheta_i(t), \ \vartheta_i^{\cdot} = \vartheta_i^{\cdot}(t), \ \vartheta_i(0) = \vartheta_{i0}, \tag{3.4}$$
$$\vartheta_i^{\cdot}(0) = \vartheta_{i0}^{\cdot}$$

where the functions $\vartheta_i = \vartheta_i(t)$ are given by the equations which are obtained from the Lagrange equations with Lagrange function (2.9) after substituting the values (3.4) in them.

Relations (3.4) describe the oscillatory motions of an n-link rod pendulum in the vertical plane. The rotatory motions of the bodies about the links are given by the quadratures (φ_{i0} are the initial values of the angles φ_i)

$$\int_{\varphi_{i0}}^{\varphi_i} \frac{d\varphi_i}{\sin\varphi_i} = -\frac{A_{13}^{i}}{A_{11}^{i}}(\vartheta_i(t) - \vartheta_{i0})$$

These two classes of motions of a chain of Hess gyroscopes, supplement the class of regular precessions of a system of Lagrange gyroscopes /5/, and thereby extend our ideas of the possible types of motions of couplings of rigid bodies.

Our results can be extended to a system of heavy bodies consisting of any number of chains of such bodies, provided that each body has not more than two spherical hinges, clamping it with other bodies or with a fixed point.

4. Let us dwell on the conditions for the first type of motions to exist. On the basis of (3.1) and (3.2), relations (2.4) can be written in the vector form

$$\boldsymbol{\omega}^{i} = \varphi_i^{\cdot}\mathbf{e}_1^{i} + \omega\mathbf{v} \tag{4.1}$$

$$\mathbf{v} = \cos\vartheta_{i0}\varepsilon_1{}^i + \sin\vartheta_{i0}\sin\varphi_i\varepsilon_2{}^i + \sin\vartheta_{i0}\cos\varphi_i\varepsilon_3{}^i$$

Here, ϑ_{i0} are the angles between the vectors $\varepsilon_1{}^i$ and ϑ, while the time-dependence of φ_i is given by (3.3). The motions for which the angular velocity vector has the form (4.1) are called semiregular precessions /6/. Let us find the conditions for motions of this type to exist of Hess gyroscopes. For this, we add to Eqs.(2.1) the equations of motion of the centre of mass

$$m_i\mathbf{v}_i^{\cdot} = -m_ig\mathbf{v} + \mathbf{R}^i - \mathbf{R}^{i+1} \tag{4.2}$$

where g is the acceleration due to gravity. Using the equations

$$d\varepsilon^i/dt = \omega(\mathbf{v}\times\varepsilon_1{}^i),\quad d^2\varepsilon_1{}^i/dt^2 = \omega^2(\cos\vartheta_{i0}\mathbf{v} - \varepsilon_1{}^i)$$

we obtain from (4.2):

$$\begin{gathered} m_i\{g\mathbf{v} + \omega^2\Big[\sum_{k=1}^{i-1}(r^k+\rho^k)(\cos\vartheta_k\mathbf{v} - \varepsilon_1{}^k) + \\ r^i(\cos\vartheta_{i0}\mathbf{v} - \varepsilon_1{}^i)\Big]\} = \mathbf{R}^i - \mathbf{R}^{i+1} \end{gathered} \tag{4.3}$$

We resolve the vector $\varepsilon_1{}^k$, $k<i$, in the basis $\varepsilon_1{}^i$ and $\mathbf{v}$

$$\varepsilon_1{}^k = [\sin(\vartheta_{i0}-\vartheta_{k0})\mathbf{v} + \sin\vartheta_{k0}\varepsilon_1{}^i]\sin^{-1}\vartheta_{i0}$$

We then have from (4.3):

$$\begin{gathered} m_i\{[g + \omega^2\cos\vartheta_{i0}(r^i+\Sigma_i)]\mathbf{v} - \omega^2(r^i+\Sigma_i)\varepsilon_1{}^i\} = \mathbf{R}^i - \mathbf{R}^{i+1} \\ \Sigma_i = \frac{1}{\sin\vartheta_{i0}}\sum_{k=1}^{i-1}(r^k+\rho^k)\sin\vartheta_{k0} \end{gathered} \tag{4.4}$$

Consider Eqs.(2.1). Assume as in /6/ that $\varepsilon_1{}^i\times\mathbf{R}^{i+1} = 0$. We introduce the notation for $R_j{}^i$:

$$\mathbf{R}^i = R_1{}^i\varepsilon_1{}^i + R_2{}^i\varepsilon_2{}^i + R_3{}^i\varepsilon_3{}^i \tag{4.5}$$

After substituting (4.1) and (4.5) into (2.1) and using (3.2), we obtain

$$R_2{}^i/\sin\varphi_i = R_3{}^i/\cos\varphi_i = -(\omega^2/r^i)A_{22}{}^i\sin\vartheta_{i0}\cos\vartheta_{i0} \tag{4.6}$$

We return to Eqs.(4.4). We project the left- and right-hand sides of (4.4) onto the vectors $\varepsilon_j{}^i$ $(j=1,2,3)$ and substitute the values (4.6) into the resulting expressions:

$$\begin{gathered} m_i[g\cos\vartheta_{i0} - \omega^2\sin^2\vartheta_{i0}(r^i+\Sigma_i)] = R_1{}^i - R_1^{i+1} \\ m_i\omega^2r^{i2}\cos\vartheta_{i0} + m_ir^i(g + \omega^2\cos\vartheta_{i0}\Sigma_i) + \omega^2A_{22}{}^i\cos\vartheta_{i0} = 0 \end{gathered} \tag{4.7}$$

Consider the second group of n equations in (4.7). We assume that it serves to define the r^i. In this case, we must have two inequalities, the first being

$$m_i(g + \omega^2\cos\vartheta_{i0}\Sigma_i)^2 \geqslant 4\omega^4A_{22}{}^i\cos^2\vartheta_{i0} \tag{4.8}$$

which serves as a limit on the parameter $A_{22}{}^i$; the second is $\cos\vartheta_{i0}<0$, which sets a limit on the angles ϑ_{i0}.

The first group of n equations in (4.7) is used to find $R_1{}^i$, $R_1{}^{i+1}$; the equations are best considered in turn, by first writing them for the n-th body $(R_1{}^{n+1}=0)$ and finding $R_1{}^n$, then finding recurrently the reactions $R_1{}^{n-1}, R_1{}^{n-2}, \ldots, R_1{}^1$.

To sum up, assuming that $\varepsilon_1{}^i\times\mathbf{R}^{i+1} = 0$, the conditions for semiregular precessions of a chain of Hess gyroscopes to exist are inequalities $\cos\vartheta_{i0}<0$, (4.8), and the second group of n equations in (4.7), which enable the parameters $r^1, r^2, \ldots, r^n$ to be found. This approach can also be used to study the conditions for the second class of motions in Sect.3 to exist.

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THE STABILITY OF THE STEADY-STATE MOTIONS OF A SYSTEM WITH PSEUDOCYCLICAL COORDINATES*

V.A. ATANASOV and L.K. LILOV

The sufficient conditions for the asymptotic stability of the steady-state motions of a mechanical system with pseudocyclical coordinates, by means of forces acting on these coordinates when dissipation with respect to the positional coordinates is present, are formulated. Both gyroscopically connected and unconnected systems are considered. The results are used to study the possible stabilization of the steady-state motion of an unbalanced rotor on a flexible shaft.

1. Consider a holonomic scleronomic mechanical system with n degrees of freedom. Let q_j be the generalized coordinates of the system, $q_j^{\cdot}$, $p_j^{\cdot}$ the generalized velocities and momenta $(j = 1, \ldots, n)$, T and π the kinetic and potential energies respectively, and $L = T - \pi$ the Lagrange function. Let non-potential forces Q_j $(j = 1, \ldots, n)$ as well as potential forces, act on the system. It will be assumed throughout that there are coordinates q_α (always, $\alpha = m+1, \ldots, n;\ m < n$) which do not appear explicitly in the expression for the Lagrange function L $(\partial L/\partial q_\alpha = 0)$. We also assume that the forces acting on the system are likewise independent of these coordinates, which we shall call pseudocyclical. The remaining coordinates q_i $(i = 1, \ldots, m)$ are positional. The generalized non-potential forces Q_i $(i = 1, \ldots, m)$ will be regarded as dissipative with respect to the generalized velocities; the dissipation may be incomplete, or, in particular, may be zero.

When there are no forces Q_α, acting on the pseudocyclical coordinates, the system can perform a steady-state motion, in which the potential coordinates q_i and the pseudocyclical velocities $q_\alpha^{\cdot}$ remain constant, while the pseudocyclical coordinates q_α vary linearly with time. Our main problem is to find the conditions under which the steady-state motion can be stabilized up to asymptotic stability with respect to the positional coordinates and all the velocities, by means of forces Q_α which act only on the pseudocyclical coordinates.

This problem was first considered in /1, 2/ when studying mechanical systems when there is no dissipation. It was proposed in /3/ to choose the forces Q_α in such a way that a pre-assigned linear manifold proved to be an invariant asymptotically stable integral manifold for the system of linearized differential equations of the perturbed motion. If the linearized system is then asymptotically stable on the manifold with respect to the positional coordinates, these forces Q_α then solve the problem of the asymptotic stability of the steady-state motion. This method of constructing the stabilizing signals was used to study the stability of any steady-state motions of gyroscopically unconnected systems /3/ and the trivial steady-state motions of gyroscopically connected systems /4/. Different methods may be used to conctruct the stabilizing signals, in particular the method given in /5/.

However, before trying to construct the stabilizing signals, we must ask the fundamental questions as to whether a given steady-state motion can in fact be stabilized by forces which act on the pseudocyclical coordinates. Below, we state sufficient conditions for this problem to be solvable for any systems with pseudocyclical coordinates when there are dissipative forces on the positional coordinates.

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